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Design and power loss evaluation of a noncircular gear pair for an infinitely variable transmission

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ABSTRACT

A novel noncircular gear design method is proposed in this work based on a modified high-order elliptical pitch curve for an infinitely variable transmission (IVT), and power loss evaluation of a noncircular gear pair is investigated. Modified high-order elliptical pitch curves of noncircular gears are developed by using kinematic equations of the IVT to eliminate cusps between every two neighboring lobes of pitch curves, and improve smoothness of pitch curves and manufacturability of the noncircular gear pair. A power loss evaluation method is developed for the noncircular gear pair by calculating sliding friction power loss of meshing tooth surfaces. The time-varying friction force of every contact position on meshing tooth surfaces of the noncircular gear pair is calculated by using their variable coefficients of friction and sliding velocities. The instantaneous sliding friction power loss of the noncircular gear pair is calculated based on time-varying friction forces and sliding velocities along contact paths of tooth profiles of the noncircular gear pair.

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Abbreviations

COF	Coefficient of friction
CVT	Continuously variable transmission
ICM	Input-control module
IVT	Infinitely variable transmission
MCM	Motion-conversion module
NG1, NG2	Driven and driving noncircular gears
PGS	Planetary gear set
SYS	Scotch yoke system

1. Introduction

Circular or cylindrical gear pairs are used in most gear transmissions for different applications, such as automotive and mining machinery [1–3]. These gear transmissions are designed to transmit power with constant speed ratios and low power

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Nomenclature

B	Face width
d_{pg}	Pitch diameter of planet gears of planetary gear sets
d_{sg}	Pitch diameter of sun gears of planetary gear sets
F_N	Normal load
F_μ	Friction force
i	Input-to-output speed ratio of the IVT
i_{cg}	Speed ratio between ring gears and crank gears
i_n	Speed ratio of the noncircular gear pair
m	Modulus of noncircular gears
p_n	Pitch of the noncircular gear pair
r_{cg}	Pitch radius of crank gears
r_d	Pitch radius of the shaping cutter
r_{NG1}, r_{NG2}	Pitch radii of driven and driving noncircular gears
$\mathbf{r}_{NG1}, \mathbf{r}_{NG2}$	Position vectors of elliptical pitch curves of driven and driving noncircular gears
r_{og}	Pitch radius of the output gear
S_{avg}	Average surface roughness
s_{y1}, s_{y2}	Displacements of yokes in the first and second scotch yoke systems
t	Time
v_{y1}, v_{y2}	Translational speeds of yokes in the first and second scotch yoke systems
v_s, v_e	Sliding velocity and entraining velocity
V_p	Inverse of the elastic potential energy
z_n	Number of teeth of the driven noncircular gear
ε_β	Face contact ratio
ε_n	Eccentricity of the elliptical pitch curve
φ_{cr}	Rotational angle of the crank
φ_n	Rotational angle of the driven noncircular gear
φ_p	Rotational angle of the prime mover
φ_{sg}	Rotational angle of the control gear
κ_n	Slope of an arbitrary point on the elliptical pitch curve of the driven noncircular gear
η_n	Number of lobes of the pitch curve of the driven noncircular gear
λ_{nj}	Modified coefficient of the j th elliptical pitch curve
μ_n	Order of the elliptical pitch curve
ρ_n	Curvature of each lobe of the pitch curve of the driven noncircular gear
σ	Viscosity of lubrication oil entering gear contact
ℓ	Length of each segment of the driven noncircular gear
ℓ_{cr}	Crank length
ζ	Coefficient of friction
ω_{cg}	Rotational speed of two crank gears
ω_{cr}	Rotational speed of the crank
ω_n	Rotational speed of the driven noncircular gear
ω_{og}	Rotational speed of output gears
ω_p	Rotational speed of the prime mover
ω_r	Rotational speeds of ring gears of planetary gear sets
ω_{sg}	Control speed of planetary gear sets
ω_u	Output speed of the IVT
$\mathcal{D}$	Center distance of two noncircular gears
$\mathcal{L}_{NG1}$	Perimeter of the elliptical pitch curve
Subscripts	
cg	Crank gears
n	Noncircular gears
og	Output gears
p	Prime mover
pg	Planet gears
r	Ring gears
sg	Sun gears
$y1, y2$	Yokes in the first and second scotch yoke systems

losses using finite numbers of gear pairs. Finite numbers of gear pairs with constant speed ratios limit ranges of speed ratios of gear transmissions, which results in low fuel efficiency [4,5]. Other gear transmissions with continuously variable speed ratios have been developed for power transfer, i.e., continuously variable transmissions (CVTs) [6], and infinitely variable transmissions (IVTs) [7–10] that can achieve a zero gear ratio. CVTs can achieve continuous ranges of speed ratios using friction-based chain-pulley systems that cause slipping power losses in low-speed and high-torque conditions [11,12]. Due to effects of noncircular gears on speed ratios and torque transfer, IVTs with noncircular gears can operate with continuously variable speed ratios for desired output speeds and transmit large torques with contact forces [8,13]. Compared with current friction-based CVT technologies, IVTs with noncircular gears have obvious advantages in low-speed and high-torque applications. Since asymmetry of a noncircular gear pair can instantaneously change the speed ratio throughout its meshing process, it is used to eliminate speed variations of an IVT. The noncircular gear pair can satisfy specific motion requirements of the displacement or rotational speed of the driven gear. Noncircular gears can also be designed to reduce undesirable speed and torque fluctuations in multi-rotor systems [14].

Elliptical gears are a common type of noncircular gears, which can be designed with variable speed ratios for arbitrary motion conversion [15]. Tooth surfaces of an elliptical gear pair can properly mesh when elliptical gears rotate about their focal points. Since pitch curves of elliptical gears are ellipses, it is difficult to design and manufacture elliptical gears. While tooth profiles of elliptical gears can be generated based on the conjugate meshing principle [1], generation motion and cutting kinematics of elliptical gears are challenging for both designers and engineers. Some generation methods of standard and nonstandard involute tooth profiles [16–18] and circular-arc tooth profiles [19–21] of elliptical gears were developed by using rack or shaper cutters. Curvature analysis methods were proposed to avoid undercutting conditions of their manufacturing processes. Base curves, pitch curves, and tooth profiles of basic and N -lobed elliptical gears were developed in [22,23] by using conjugate shaper cutters with involute tooth profiles. Li et al. [24] developed a simplified numerical tooth profile generation method of noncircular gears by using a gear shaping process without solving complicated meshing equations. A high-order elliptical gear pump was developed to eliminate flow pulsation [25]. Pitch curves of high-order elliptical gears were designed based on average flow rates of these pumps. An indexing mechanism that comprises noncircular planetary gear drives was proposed to obtain a high contact ratio [26]. Since Bézier curves can be used to identify local geometric features of gear tooth surfaces [27], a conjugate pitch curve modeling method was developed to define angular coordinates of pitch curves of N -lobed elliptical gears based on Bézier curves [28]. Tooth profiles of noncircular gears are manufactured by hobbing and shaping machine tools with rack and shaper cutters, respectively, which are similar to manufacturing processes of circular gears [29]. Due to complex geometric features of noncircular gears, more analysis of tooth profiles of noncircular gears is required to optimize gear parameters to achieve better meshing performance. Pitch curves must be designed to achieve desired velocity profiles of variable motions of noncircular gear pairs.

Much research has been devoted to pitch curve design and tooth profile manufacturing of noncircular gears. However, smoothness of pitch curves and efficiency of noncircular gears have not been clarified yet. Since noncircular gears rotate with non-uniform speeds, elliptical gears demonstrate particular characteristics of non-smooth pitch curves and low efficiency, which can increase dynamic friction forces [30] and failure rates of gear transmissions [31]. While a previous design of a noncircular gear pair can eliminate speed variations of an IVT [8], there are cusps between two neighboring lobes of pitch curves of noncircular gears, which reduce smoothness of pitch curves and increase sliding friction forces of tooth profiles. Additionally, it is difficult to manufacture tooth profiles at or near cusps of pitch curves of noncircular gears by rack or shaper cutters. Principal power losses of gear transmissions are caused by sliding friction of meshing gear tooth surfaces, churning of lubrication oil, and friction inside shaft support bearings [32–34]. Gear meshing power loss has dominant impacts on overall performance of a gear pair in low-speed and high-torque applications [35–37].

This work presents novel modified high-order elliptical pitch curves of noncircular gears for the IVT in [8] to improve smoothness of pitch curves and manufacturability of noncircular gears. The modified high-order elliptical pitch curves are designed based on kinematic equations of the IVT. Power loss of the noncircular gear pair is evaluated by incorporating load-dependent friction forces and sliding velocities of meshing tooth surfaces. A power loss evaluation method of noncircular gears is developed for changing speed and load conditions. Particular meshing characteristics of tooth surfaces of noncircular gears with modified high-order elliptical pitch curves are determined, and time-varying load distribution of tooth surfaces of noncircular gears is considered in the calculation process of tooth surface friction forces. Key merits of the proposed noncircular gear design method with a modified high-order elliptical pitch curve are: (1) the resulting IVT can improve its meshing performance with large speed variation that is induced by the variable input speed; and (2) power loss evaluation of noncircular gears with modified high-order elliptical pitch curves can reduce their dynamic friction forces.

The remaining part of this paper is organized as follows: the kinematic model of the IVT is introduced in Section 2. High-order elliptical pitch curves of noncircular gears are developed in Section 3.1 based on the kinematic model of the IVT. Convexity analysis of high-order elliptical pitch curves is presented in Section 3.2. Modified high-order elliptical pitch curves are proposed in Section 3.3 to improve manufacturability of noncircular gears. Tooth profiles of noncircular gears are determined by modified high-order elliptical pitch curves in Section 4. The power loss evaluation method of noncircular gears and numerical simulation results of sliding friction power losses are presented in Section 5. Finally, some conclusions from this study are presented in Section 6.

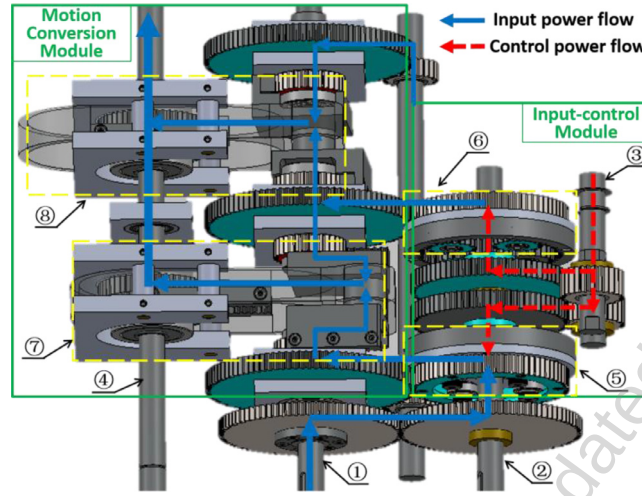


Fig. 1. Schematic of the IVT in [8], where ① is the input shaft, ② is the secondary shaft, ③ is the control shaft, ④ is the output shaft, ⑤ is the first PGS, ⑥ is the second PGS, ⑦ is the first SYS, and ⑧ is the second SYS.

2. Layout and kinematic model of the IVT

The basic principle of the IVT in [8] is to mechanically transmit a variable input speed to a desired constant output speed with a continuously variable speed ratio. The continuously variable speed ratio of the IVT can be adjusted by changing the crank length in scotch yoke systems (SYSs). The noncircular gear pair is designed to eliminate speed variations of the output speed of the IVT.

2.1. Layout and overview of the IVT

The IVT in [8] in Fig. 1 consists of a noncircular gear pair, an input-control module (ICM), and a motion-conversion module (MCM). An input rotational speed is first transmitted from the noncircular gear pair to the ICM. Driving and driven noncircular gears are installed on the input shaft and secondary shaft, respectively. With the modulation effect of the noncircular gear pair, a modulated input speed is translated to the secondary shaft in the ICM. The stepper motor mounted on the control shaft in the ICM is used to provide a control speed for adjustment of the input-to-output speed ratio of the IVT.

The control speed and modulated input speed are combined by two 2K-H planetary gear sets (PGSs) on the secondary shaft in the ICM. Combined speeds of ring gears of two PGSs are transmitted to SYSs in the MCM. Two SYSs transmit the combined speeds to translational speeds of yokes; four meshed rack-pinion sets then convert translational speeds of yokes to rotational speeds of four output gears. The output speed of the IVT is the maximum rotational speed of output gears that are rectified by one-way bearings. The detailed design of the IVT is described in [8].

2.2. Kinematic model of the IVT

Noncircular gears used in the IVT are denoted as NG1 and NG2 for the driven noncircular gear on the secondary shaft in the MCM and driving noncircular gear on the input shaft, respectively. The input rotational speed ω_p is transmitted from the prime mover to NG2, as shown in Fig. 1. The rotational speed ω_n of NG1 is transmitted to the corresponding sun gears of two PGSs through the secondary shaft. The control speed ω_{sg} of PGSs are transmitted from the stepper motor to sun gears of PGSs. The rotational speed ω_n of NG1 and control speed ω_{sg} of PGSs are combined with rotational speeds of ring gears. Rotational speeds of ring gears of two PGSs can be represented as

$$\omega_r = \frac{2(d_{sg} + d_{pg})\omega_n - d_{sg}\omega_{sg}}{d_{sg} + 2d_{pg}} \quad (1)$$

where d_{sg} is the pitch diameter of sun gears of PGSs, and d_{pg} is the pitch diameter of planet gears.

Rotational speeds ω_r of ring gears of PGSs are transmitted to two crank gears of the first and second SYSs in the MCM. Rotational speeds of two crank gears can be represented as $\omega_{cg} = -i_{cg}\omega_r$, where i_{cg} is the speed ratio between ring gears and crank gears in SYSs. Here, the speed ratio i_{cg} is designed as 3/4. With the rotational speed of the crank ω_{cr} , where $\omega_{cr} = \omega_{cg}$, a yoke is driven to move with a translational speed in the first SYS. The angle between two cranks in the first and second SYSs is 90°. Displacements of yokes in the first and second SYSs can be represent as

$$s_{y1} = \ell_{cr} \cos \varphi_{cr} \quad (2)$$

$$s_{y2} = \ell_{cr} \cos(\varphi_{cr} - 90^\circ) \quad (3)$$

respectively, where ℓ_{cr} is the crank length, and φ_{cr} is the rotational angle of the crank that can be represented as $\varphi_{cr} = \int \omega_{cr} dt$. Translational speeds of yokes in the first and second SYSs can be represented as

$$v_{y1} = \dot{s}_{y1} = \dot{\ell}_{cr} \cos \varphi_{cr} - \ell_{cr} \omega_{cr} \sin \varphi_{cr} \quad (4)$$

$$v_{y2} = \dot{s}_{y2} = \dot{\ell}_{cr} \cos(\varphi_{cr} - 90^\circ) - \ell_{cr} \omega_{cr} \sin(\varphi_{cr} - 90^\circ) \quad (5)$$

respectively. The yokes are driven by cranks, and their translational speeds are determined by the crank length ℓ_{cr} and rotational angle φ_{cr} of the crank. The crank length ℓ_{cr} changes with rotational angles of two control gears. The crank length ℓ_{cr} can be represented as

$$\ell_{cr} = \frac{i_{cg} d_{sg} \varphi_{sg}}{d_{sg} + 2d_{pg}} r_{cg} \quad (6)$$

where r_{cg} is the pitch radius of the crank gear, and φ_{sg} is the rotational angle of the corresponding control gear that can be obtained by integrating the control speed ω_{sg} with respect to time. The translational speeds of the yokes can be represented as

$$v_{y1} = -\ell_{cr} \omega_{cr} \sin \varphi_{cr} \quad (7)$$

$$v_{y2} = -\ell_{cr} \omega_{cr} \sin(\varphi_{cr} - 90^\circ) \quad (8)$$

respectively. Four rack-pinion sets are used here for translational-rotational motion conversion. Rotational speeds of output gears for two SYSs can be represented as

$$\omega_{og} = \begin{bmatrix} \omega_{og1} \\ \omega_{og2} \\ \omega_{og3} \\ \omega_{og4} \end{bmatrix} = \begin{bmatrix} v_{y1}/r_{og} \\ -v_{y1}/r_{og} \\ v_{y2}/r_{og} \\ -v_{y2}/r_{og} \end{bmatrix} = \frac{\ell_{cr} \omega_{cr}}{r_{og}} \begin{bmatrix} \sin(\varphi_{cr} - 180^\circ) \\ \sin \varphi_{cr} \\ \sin(\varphi_{cr} - 270^\circ) \\ \sin(\varphi_{cr} - 90^\circ) \end{bmatrix} \quad (9)$$

where r_{og} is the pitch radius of the output gear, and the "-" sign before v_{y1} and v_{y2} means the counterclockwise direction of the output shaft, as shown in Fig. 1. Since four one-way bearings only allow the maximum rotational speed of four output gears in the clockwise direction to transmit to the output shaft of the IVT, the output speed of the IVT can be represented as

$$\omega_u = \frac{\ell_{cr} \omega_{cr}}{r_{og}} \max \{ \sin \varphi_{cr}, \sin(\varphi_{cr} - 90^\circ), \sin(\varphi_{cr} - 180^\circ), \sin(\varphi_{cr} - 270^\circ) \} \quad (10)$$

where $\max \{ \cdot \}$ is a function that selects the maximum value of its arguments. Since sun gears and planet gears of PGs are designed with $d_{sg} = d_{pg}$, Eq. (1) can also be written as

$$\omega_r = \frac{4}{3} \omega_n - \frac{1}{3} \omega_{sg} \quad (11)$$

With the same initial installation position of SYSs, the relationship between the rotational angle of the crank φ_{cr} and that of NG1 φ_n is $\varphi_{cr} = \varphi_n$. Hence, the input-to-output speed ratio of the IVT can be represented as

$$i = \frac{\omega_p}{\omega_u} = \frac{\sqrt{2} \pi r_{og}}{4 \ell_{cr}} \quad (12)$$

Since r_{og} is constant, the input-to-output speed ratio of the IVT can be determined by ℓ_{cr} .

3. High-order pitch curve design

The noncircular gear pair is designed to transmit power between the input shaft and secondary shaft of the IVT, which has a variable speed ratio. Synthesis of high-order elliptical pitch curves of noncircular gears is designed by formulation of a convexity condition through determination of their speed ratio and rotation transformation of centroids. In this section, determination of conjugated high-order elliptical pitch curves is investigated based on kinematic equations of the IVT developed in Section 2. This analysis can be used to choose a suitable shaper-cutter for manufacturing of noncircular gears.

3.1. High-order elliptical pitch curves

Noncircular gears are designed in this study based on high-order elliptical pitch curves instead of typical elliptically-shaped pitch curves. Since the input-to-output speed ratio of the IVT depends on crank lengths of SYSs, the changing rate

of the crank length causes instantaneous speed variations of the output speed of the IVT, which result in variations of the instantaneous speed ratio of the MCM. The speed ratio between the ICM and MCM can be represented as

$$i_n(\varphi_n) = \frac{\omega_p(\varphi_p)}{\omega_n(\varphi_n)} = \frac{d\varphi_p/dt}{d\varphi_n/dt} \quad (13)$$

where φ_n is the rotational angle of NG1 with a period of 2π , ω_n is the rotational speed of NG1, and φ_p is the rotational angle of NG2. The speed ratio between the output shaft and NG2 can be represented as

$$\frac{i(\varphi_n)}{i_n(\varphi_n)} = \frac{\omega_u(\varphi_n)}{\omega_n(\varphi_n)} \frac{\omega_n(\varphi_n)}{\omega_p(\varphi_p)} = \frac{\omega_u(\varphi_n)}{\omega_p(\varphi_p)} \quad (14)$$

By submitting Eq. (10) and $\varphi_{cr} = \varphi_n$ into Eq. (14), the speed ratio of the noncircular gear pair can be represented as

$$i_n(\varphi_n) = \frac{\sqrt{2}\pi}{4} \max \{ \sin \varphi_n, \sin (\varphi_n - 90^\circ), \sin (\varphi_n - 180^\circ), \sin (\varphi_n - 270^\circ) \} \quad (15)$$

The speed ratio $i_n(\varphi_n)$ that is obtained by Eq. (15) can be applied to design pitch curves of the conjugated noncircular gear pair of the IVT. The center distance of two noncircular gears is

$$\mathcal{D} = r_{NG1}(\varphi_n) + r_{NG2}(\varphi_p) \quad (16)$$

where $r_{NG1}(\varphi_n)$ and $r_{NG2}(\varphi_p)$ are pitch radii of NG1 and NG2 when their rotational angles are φ_n and φ_p , respectively. According to Eq. (13), the differential relation between rotational angles φ_p and φ_n is

$$d\varphi_p = i_n(\varphi_n) d\varphi_n \quad (17)$$

The position vector of the high-order elliptical pitch curve of NG1 can be represented as

$$\mathbf{r}_{NG1}(\varphi_n) = \begin{cases} \frac{p_n}{1 - \varepsilon_n \cos \left[\mu_n \lambda_{n1} \left(\varphi_n - \frac{2\pi}{\mu_n} \right) \right]} & \left(0 \leq \varphi_n \leq \frac{2\pi}{\mu_n} + \frac{2\pi}{\eta_n \lambda_{n1}} \right) \\ \frac{p_n}{1 - \varepsilon_n \cos \left[\mu_n \lambda_{n2} \left(\varphi_n - \frac{2\pi}{\mu_n} - \frac{2\pi}{\eta_n \lambda_{n1}} \right) + \frac{2\pi}{\eta_n} \right]} & \left(\frac{2\pi}{\mu_n} + \frac{2\pi}{\eta_n \lambda_{n1}} < \varphi_n \leq \frac{2\pi}{\mu_n} + \frac{2\pi}{\eta_n \lambda_{n1}} + \frac{2\pi}{\eta_n \lambda_{n2}} \right) \\ \frac{p_n}{1 - \varepsilon_n \cos \left[\mu_n \lambda_{nj} \left(\varphi_n - \frac{2\pi}{\mu_n} - \sum_{j=1}^{j-1} \frac{2\pi}{\eta_n \lambda_{nj}} \right) + \frac{2(j-1)\pi}{\eta_n} \right]} & \left(\frac{2\pi}{\mu_n} + \sum_{j=1}^{j-1} \frac{2\pi}{\eta_n \lambda_{nj}} < \varphi_n \leq \frac{2\pi}{\mu_n} + \sum_{j=1}^j \frac{2\pi}{\eta_n \lambda_{nj}} \right) \end{cases} \quad (18)$$

where $j = 1, 2, \dots, \eta_n$, η_n is the number of lobes of the pitch curve of NG1, μ_n is the order of the elliptical pitch curve, ε_n is the eccentricity of the elliptical pitch curve, λ_{nj} is the modified coefficient of the j th elliptical pitch curve, and p_n is the pitch of the noncircular gear pair that can be represented as

$$p_n = \frac{\pi m z_n}{\int_0^{2\pi} \frac{\sqrt{1 + \varepsilon_n^2 \sin^2 \varphi_n}}{1 + \varepsilon_n \cos \varphi_n} d\varphi_n} \quad (19)$$

in which m is the modulus of NG1 and NG2, and z_n is the number of teeth of NG1. For the high-order elliptical pitch curve of NG1, each modified coefficient λ_{nj} should satisfy $\lambda_{nj} > 1/\eta_n$ and

$$\sum_{j=1}^{\eta_n} \frac{1}{\lambda_{nj}} = \eta_n \quad (20)$$

If $\varphi_n = 0$ or 2π , one has $|\mathbf{r}_{NG1}| = p_n$. Hence, the pitch curve of NG1 that is obtained by Eq. (18) is continuous at the cusp between two neighboring lobes of the pitch curve. In order to evenly distribute teeth on the high-order elliptical pitch curve of NG1, the perimeter of the high-order elliptical pitch curve satisfies

$$\mathcal{L}_{NG1} = \int_0^{2\pi} \sqrt{|\mathbf{r}_{NG1}|^2 + \left(\frac{d\mathbf{r}_{NG1}}{d\varphi_n} \right)^2} d\varphi_n = m\pi z_n \quad (21)$$

According to Eqs. (16)–(18), pitch curves of NG1 and NG2 are conjugated with each other, and the speed ratio between NG1 and NG2 can also be represented as

$$i_n(\varphi_n) = \begin{cases} \frac{D - p_n - a\varepsilon_n \cos(\lambda_{n1}\varphi_n)}{p_n} & \left(0 \leq \varphi_n \leq \frac{2\pi}{\eta_n\lambda_{n1}}\right) \\ \frac{D - p_n - a\varepsilon_n \cos\left[\lambda_{n2}\left(\varphi_n - \frac{2\pi}{\eta_n\lambda_{n1}}\right) + \frac{2\pi}{\eta_n}\right]}{p_n} & \left(\frac{2\pi}{\eta_n\lambda_{n1}} < \varphi_n \leq \frac{2\pi}{\eta_n\lambda_{n1}} + \frac{2\pi}{\eta_n\lambda_{n2}}\right) \\ \frac{D - p_n - a\varepsilon_n \cos\left[\lambda_{nj}\left(\varphi_n - \sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}\right) + \frac{2(j-1)\pi}{\eta_n}\right]}{p_n} & \left(\sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}} < \varphi_n \leq \sum_{j=1}^j \frac{2\pi}{\eta_n\lambda_{nj}}\right) \end{cases} \quad (22)$$

Since the pitch curve of NG2 is a closed curve, the speed ratio $i_n(\varphi_n)$ satisfies

$$\int_0^{2\pi} \frac{1}{i_n(\varphi_n)} d\varphi_n = \int_0^{\frac{2\pi}{\eta_n\lambda_{n1}}} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos(\lambda_{n1}\varphi_n)} d\varphi_n + \sum_{j=2}^{\eta_n} \int_{\sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}}^{\sum_{j=1}^j \frac{2\pi}{\eta_n\lambda_{nj}}} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos\left[\lambda_{nj}\left(\varphi_n - \sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}\right) + \frac{2(j-1)\pi}{\eta_n}\right]} d\varphi_n = 2\pi \quad (23)$$

According to Eqs. (17) and (23), the rotational angle of NG2 can be written as

$$\varphi_n = \begin{cases} \int_0^{\varphi_n} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos(\lambda_{n1}\varphi_n)} d\varphi_n & \left(0 \leq \varphi_n \leq \frac{2\pi}{\eta_n\lambda_{n1}}\right) \\ \int_0^{\frac{2\pi}{\eta_n\lambda_{n1}}} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos(\lambda_{n1}\varphi_n)} d\varphi_n + \int_{\frac{2\pi}{\eta_n\lambda_{n1}}}^{\varphi_n} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos\left[\lambda_{n2}\left(\varphi_n - \frac{2\pi}{\eta_n\lambda_{n1}}\right) + \frac{2\pi}{\eta_n}\right]} d\varphi_n & \left(\frac{2\pi}{\eta_n\lambda_{n1}} < \varphi_n \leq \frac{2\pi}{\eta_n\lambda_{n1}} + \frac{2\pi}{\eta_n\lambda_{n2}}\right) \\ \int_0^{\frac{2\pi}{\eta_n\lambda_{n1}}} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos(\lambda_{n1}\varphi_n)} d\varphi_n + \sum_{j=2}^{\eta_n-1} \int_{\sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}}^{\sum_{j=1}^j \frac{2\pi}{\eta_n\lambda_{nj}}} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos\left[\lambda_{nj}\left(\varphi_n - \sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}\right) + \frac{2\pi}{\eta_n}\right]} d\varphi_n & \left(\sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}} < \varphi_n \leq \sum_{j=1}^j \frac{2\pi}{\eta_n\lambda_{nj}}\right) \\ + \sum_{j=2}^{\eta_n-1} \int_{\sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}}^{\varphi_n} \frac{p_n}{D - p_n - \mathcal{D}\varepsilon_n \cos\left[\lambda_{nj}\left(\varphi_n - \sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}\right) + \frac{2(j-1)\pi}{\eta_n}\right]} d\varphi_n & \end{cases} \quad (24)$$

The center distance $\mathcal{D}$ of two noncircular gears is equal to the constant center distance between the input shaft and secondary shaft of the IVT. According to Eqs. (16), (18), and (24), the position vector of the high-order elliptical pitch curve of NG2 can be represented as

$$\mathbf{r}_{NG2}(\varphi_n) = \begin{cases} \begin{cases} D - \frac{p_n}{1 - \varepsilon_n \cos(\lambda_{n1}\varphi_n)} \\ D - \frac{p_n}{1 - \varepsilon_n \cos\left[\mu_n\lambda_{n2}\left(\varphi_n - \frac{2\pi}{\mu_n} - \frac{2\pi}{\eta_n\lambda_{n1}}\right) + \frac{2\pi}{\eta_n}\right]} \\ D - \frac{p_n}{1 - \varepsilon_n \cos\left[\mu_n\lambda_{nj}\left(\varphi_n - \frac{2\pi}{\mu_n} - \sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}}\right) + \frac{2(j-1)\pi}{\eta_n}\right]} \end{cases} & \begin{cases} \left(0 \leq \varphi_n \leq \frac{2\pi}{\eta_n\lambda_{n1}}\right) \\ \left(\frac{2\pi}{\eta_n\lambda_{n1}} < \varphi_n \leq \frac{2\pi}{\eta_n\lambda_{n1}} + \frac{2\pi}{\eta_n\lambda_{n2}}\right) \\ \left(\sum_{j=1}^{j-1} \frac{2\pi}{\eta_n\lambda_{nj}} < \varphi_n \leq \sum_{j=1}^j \frac{2\pi}{\eta_n\lambda_{nj}}\right) \end{cases} \end{cases} \quad (25)$$

Table 1 lists parameters of the noncircular gear pair of the IVT. High-order elliptical pitch curves of NG1 and NG2 are generated based on Eqs. (18) and (25), as shown in Fig. 2(a) and (b), respectively. The noncircular gear pair can provide the desired speed ratio to eliminate speed variations of the IVT. There are cusps between every two neighboring lobes of high-order elliptical pitch curves of NG1 and NG2. These cusps have negative effects on meshing smoothness and manufacturability of the noncircular gear pair. For noncircular gear manufacturing, all lobes of high-order elliptical pitch curves of noncircular gears should be convex. Convexity analysis of high-order elliptical pitch curves of NG1 and NG2 is used for determining the eccentricity ε_n of the noncircular gear pair.

3.2. Convexity analysis of high-order elliptical pitch curves

The first and second derivatives of the position vector of the high-order elliptical pitch curve of NG1 with respect to the rotational angle φ_n can be represented as

Table 1
Parameters of the noncircular gear pair.

Item	NG1	NG2
Number of teeth	42	42
Module (mm)	3	
Center distance (mm)	135	
Face width (mm)	20	
Number of lobes of the pitch curve	4	
Order of the pitch curve	4	
Eccentricity of the elliptical pitch curve	0.15	
Modified coefficient of the pitch curve	0.55	

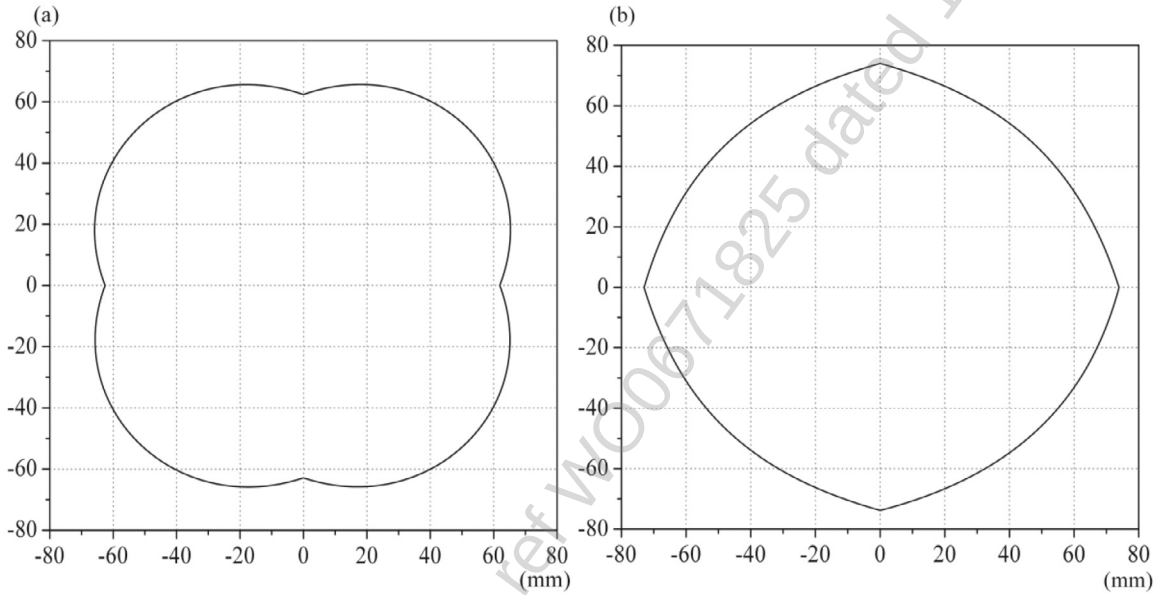


Fig. 2. High-order elliptical pitch curves of (a) NG1 and (b) NG2.

$$\dot{\mathbf{r}}_{NG1} = \frac{d\mathbf{r}_{NG1}}{d\varphi_n} = \frac{\mu_n \lambda_{nj} p_n \varepsilon_n \sin \left[\mu_n \lambda_{nj} \left(\varphi_n - \frac{2\pi}{\mu_n} \right) \right]}{\left\{ 1 - \varepsilon_n \cos \left[\mu_n \lambda_{nj} \left(\varphi_n - \frac{2\pi}{\mu_n} \right) \right] \right\}^2} \quad (26)$$

$$\ddot{\mathbf{r}}_{NG1} = \frac{d^2 \mathbf{r}_{NG1}}{d\varphi_n^2} = \frac{\mu_n^2 \lambda_{nj}^2 p_n \varepsilon_n^2 \left\{ \frac{1}{\varepsilon_n} \cos \left[\mu_n \lambda_{nj} \left(\varphi_n - \frac{2\pi}{\mu_n} \right) \right] - \sin^2 \left[\mu_n \lambda_{nj} \left(\varphi_n - \frac{2\pi}{\mu_n} \right) \right] - 1 \right\}}{\left\{ 1 - \varepsilon_n \cos \left[\mu_n \lambda_{nj} \left(\varphi_n - \frac{2\pi}{\mu_n} \right) \right] \right\}^3} \quad (27)$$

respectively. The curvature of each lobe of the pitch curve of NG1 is

$$\rho_n = p_n \frac{\left\{ \left[1 - \varepsilon_n \cos (\lambda_{nj} \varphi_n) \right]^2 + \varepsilon_n^2 \lambda_{nj}^2 \sin^2 (\lambda_{nj} \varphi_n) \right\}^{\frac{3}{2}}}{\left[1 - \varepsilon_n \cos (\lambda_{nj} \varphi_n) \right]^3 \left[1 + \varepsilon_n (\lambda_{nj}^2 - 1) \cos (\lambda_{nj} \varphi_n) \right]} \quad (28)$$

In order to make each segment of the pitch curve convex, the curvature of each lobe of the pitch curve of NG1 must be positive. Since the numerator in Eq. (28) is always greater than zero, the denominator in Eq. (28) should also be greater than zero to ensure the curvature ρ_n of each lobe of the pitch curve of NG1 be positive. Hence, each lobe of the pitch curve is convex, and one has

$$1 + \varepsilon_n (\mu_n^2 \lambda_{n1}^2 - 1) \cos (\mu_n \lambda_{n1} \varphi_n) \geq 0 \quad (29)$$

According to Eq. (29), the eccentricity ε_n of the noncircular gear pair satisfies

$$\varepsilon_n \leq \frac{1}{(1 - \mu_n^2 \lambda_{n1}^2) \cos (\mu_n \lambda_{n1} \varphi_n)} \quad (30)$$

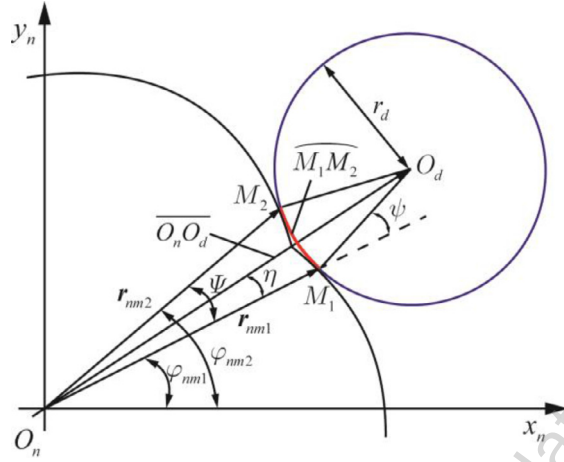


Fig. 3. Modification of the high-order elliptical pitch curve of NG1.

3.3. Modification of high-order elliptical pitch curves

Since there are cusps between every two neighboring lobes of high-order elliptical pitch curves of NG1 and NG2, the positive convexity of each lobe of the pitch curves cannot guarantee the positive convexity of the whole high-order elliptical pitch curves. Regardless of a hobbing cutter or a shaping cutter, a generation process of a noncircular gear can be described as non-sliding motion between the high-order elliptical pitch curve of NG1 and pitch circle of the hobbing or shaping cutter. There are two tangent points on both sides of a cusp between the high-order elliptical pitch curve of NG1 and pitch circle of the hobbing or shaping cutter. Hence, tooth surfaces of noncircular gears at cusps cannot be manufactured by the hobbing or shaping cutter. The high-order pitch curve modification method of noncircular gears is developed to improve manufacturability of high-order elliptical pitch curves of NG1 and NG2.

The slope of an arbitrary point on the high-order elliptical pitch curve of NG1 can be represented as

$$\kappa_n = \frac{\dot{\mathbf{r}}_{NG1} \sin \varphi_n + \mathbf{r}_{NG1} \cos \varphi_n}{\dot{\mathbf{r}}_{NG1} \cos \varphi_n - \mathbf{r}_{NG1} \sin \varphi_n} \quad (31)$$

The high-order elliptical pitch curve of NG1 and pitch circle of the shaping cutter are tangent at points M_1 and M_2 , as shown in Fig. 3. A coordinate system $S_n(x_n, y_n)$ is attached to NG1, where O_n is the center of the pitch curve of NG1. The point O_d is the center of the shaping cutter. Position vectors of M_1 and M_2 are denoted as $\mathbf{r}_{nm1}(\varphi_{nm1})$ and $\mathbf{r}_{nm2}(\varphi_{nm2})$, respectively, and φ_{nm1} and φ_{nm2} are polar angles of M_1 and M_2 , respectively. In $\triangle O M_1 M_2$, the distance between M_1 and M_2 can be represented as

$$|M_1 M_2| = \sqrt{|\mathbf{r}_{nm1}|^2 + |\mathbf{r}_{nm2}|^2 - 2|\mathbf{r}_{nm1}||\mathbf{r}_{nm2}|\cos \Psi} \quad (32)$$

where $\Psi = \varphi_{nm2} - \varphi_{nm1}$. The angle between $\mathbf{r}_{nm1}(\varphi_{nm1})$ and $\overline{O_n O_d}$ can be represented as

$$\eta = \arccos\left(\frac{|M_1 M_2|}{2r_d}\right) + \arcsin\left(\frac{|\mathbf{r}_{nm1}|\sin \Psi}{|M_1 M_2|}\right) \quad (33)$$

where r_d is the pitch radius of the shaping cutter, which is 30 mm in this study. The distance between O_n and O_d is

$$|\overline{O_n O_d}| = \sqrt{|\mathbf{r}_{nm1}|^2 + r_d^2 - 2|\mathbf{r}_{nm1}|r_d \cos \eta} \quad (34)$$

The angle between $\mathbf{r}_{nm1}(\varphi_{nm1})$ and $\overline{M_1 O_d}$ can be represented as $\psi = \arcsin\left(\frac{r_d}{|\overline{O_n O_d}|} \sin \eta\right)$. Consider an arbitrary point G_1 on the modified pitch curve $\widehat{M_1 M_2}$; the position vector of G_1 is denoted as $\mathbf{r}_{ng1}(\varphi_{ng1})$, where φ_{ng1} is the angle between $\mathbf{r}_{nm1}(\varphi_{nm1})$ and $\overline{O_n G_1}$. The distance between O_n and G_1 can be represented as

$$|\mathbf{r}_{ng1}(\varphi_{ng1})| = |\overline{O_n O_d}| \cos(\varphi_{ng1} - \psi) - \sqrt{r_d^2 - |\overline{O_n O_d}|^2 \sin^2(\varphi_{ng1} - \psi)} \quad (35)$$

The first and second derivatives of the position vector $\mathbf{r}_{ng1}$ of the point G_1 with respect to the angle φ_{ng1} can be represented as

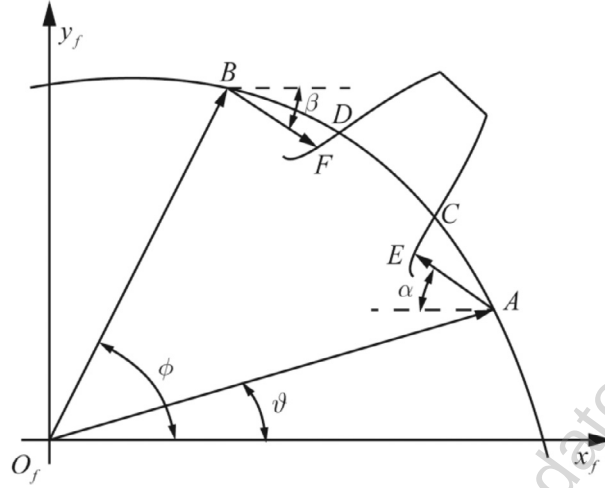


Fig. 4. Coordinates of two tooth profiles of a tooth of NG1.

$$\frac{d\mathbf{r}_{ng1}}{d\varphi_{ng1}} = \overline{O_n O_d} \left\{ \frac{\overline{O_n O_d} \cos(\varphi_{ng1} - \psi)}{\left[r_d^2 - |\overline{O_n O_d}|^2 \sin^2(\varphi_{ng1} - \psi) \right]^{1/2}} - 1 \right\} \sin(\varphi_{ng1} - \psi) \quad (36)$$

$$\frac{d^2\mathbf{r}_{ng1}}{d\varphi_{ng1}^2} = \overline{O_n O_d} \left\{ \frac{\left(|\overline{O_n O_d}|^2 - r_d^2 \right) \overline{O_n O_d} \sin^2(\varphi_{ng1} - \psi)}{\left[r_d^2 - |\overline{O_n O_d}|^2 \sin^2(\varphi_{ng1} - \psi) \right]^{3/2}} + \frac{\overline{O_n O_d} \cos^2(\varphi_{ng1} - \psi)}{\left[r_d^2 - |\overline{O_n O_d}|^2 \sin^2(\varphi_{ng1} - \psi) \right]^{1/2}} - \cos(\varphi_{ng1} - \psi) \right\} \quad (37)$$

respectively. Assume that the point G_2 is the meshing point of G_1 on the modified pitch curve of NG2. The polar angle of G_2 is denoted as φ_{2g} . The position vector of G_2 on the modified pitch curve of NG2 can be represented as

$$\mathbf{r}_{pg2} = \int_0^{2\pi} \frac{(\mathbf{r}_{ng1} - \mathbf{D})}{\mathbf{r}_{ng1}} \frac{d\mathbf{r}_{ng1}}{d\varphi_{ng1}} d\varphi_{pg2} \quad (38)$$

4. Tooth profiles of noncircular gears

For involute tooth profiles of noncircular gears, each point on a tooth profile is determined by its center of curvature. The involute tooth profile can be generated by its sequence of centers of curvature that is its evolute. Since two tooth profiles of a tooth of a noncircular gear have different evolutes, tooth profiles of the noncircular gear are represented by different involutes.

A coordinate system $S_f(x_f, y_f)$ is attached to NG1, where O_f is a focal point of the elliptical pitch curve of NG1, and ϑ and ϕ are polar angles of arbitrary points A and B on the pitch curve of NG1, respectively, as shown in Fig. 4. Points E and F are the corresponding points on right and left tooth profiles of a tooth of NG1 that are generated by points A and B, respectively. Points C and D are pitch points of right and left tooth profiles of the tooth of NG1, respectively.

The position vector of the point E can be represented as

$$\mathbf{r}_{fR} = \overline{O_f A} + \overline{AE} \quad (39)$$

Coordinates of the point E can be written as

$$\begin{cases} x_{fR} = \overline{O_f A} \cdot \cos \vartheta - \overline{AE} \cdot \cos \alpha \\ y_{fR} = \overline{O_f A} \cdot \sin \vartheta + \overline{AE} \cdot \sin \alpha \end{cases} \quad (40)$$

where α is the angle between $\overline{AE}$ and the x_f -axis. The position vector of the point F can be represented as

$$\mathbf{r}_{fL} = \overline{O_f B} + \overline{BF} \quad (41)$$

Coordinates of the point F are

$$\begin{cases} x_{fL} = \overline{O_f B} \cdot \cos \phi - \overline{BF} \cdot \cos \beta \\ y_{fL} = \overline{O_f B} \cdot \sin \phi + \overline{BF} \cdot \sin \beta \end{cases} \quad (42)$$

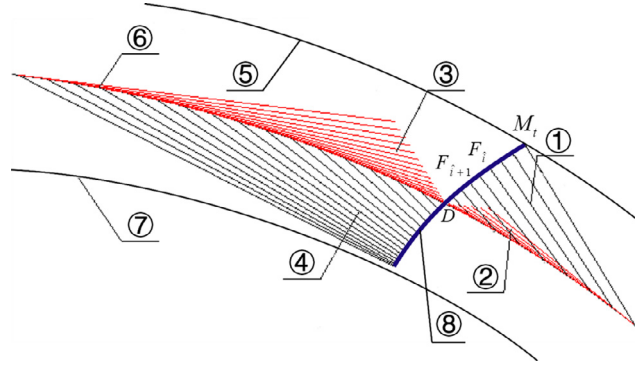


Fig. 5. Left tooth profile of a tooth of NG1, where ① and ④ are clusters of $\overline{BF}$, ② and ③ are clusters of tangent lines of the pitch curve through the movable point B , ⑤ is the tip line, ⑥ is the pitch curve, ⑦ is the root line, and ⑧ is the left tooth profile.

where β is the angle between $\overline{BF}$ and the x_f -axis. Surface coordinates of the point F are denoted as (u_F, φ_F) . The point $M_t(u_t, \varphi_t)$ is the intersection of the left tooth profile and tip line of the tooth of NG1, as shown in Fig. 5. If points F and M_t coincide, the curve length between M_t and the tip line of the tooth surface of NG1 is

$$\mathcal{F}(u_t, \varphi_t) = \mathcal{F}(u_F, \varphi_F) = 0 \quad (43)$$

The polar angle difference between F and M_t of the tooth surface of NG1 is

$$\mathcal{R}(u_t, \varphi_t) = \phi_t(u_t, \varphi_t) - \phi_F(u_F, \varphi_F) = 0 \quad (44)$$

With Eqs. (42)–(44), surface coordinates of $M_t(u_t, \varphi_t)$ can be solved using Newton-Raphson method. Consider a point $F_i(u_{F_i}, \varphi_{F_i})$ is the i th point that is generated by the movable point B on the pitch curve of NG1. Let the curve length between F_i and M_t be $\mathcal{F}(u_{F_i}, \varphi_{F_i})$, and the polar angle difference between F_i and M_t be $\mathcal{R}(u_{F_i}, \varphi_{F_i})$. Surface coordinates of F_{i+1} can be obtained by selecting Δu and $\Delta \varphi$ as increments of u_{F_i} and φ_{F_i} , respectively. Hence, $\mathcal{F}(u_{F_i} + \Delta u, \varphi_{F_i} + \Delta \varphi)$ and $\mathcal{R}(u_{F_i} + \Delta u, \varphi_{F_i} + \Delta \varphi)$ at F_{i+1} can be written as

$$\mathcal{F}(u_{F_i} + \Delta u, \varphi_{F_i} + \Delta \varphi) = \mathcal{F}(u_{F_i}, \varphi_{F_i}) + \Delta u \left(\frac{\partial \mathcal{F}}{\partial u_{F_i}} \right) + \Delta \varphi \left(\frac{\partial \mathcal{F}}{\partial \varphi_{F_i}} \right) + \dots = 0 \quad (45)$$

$$\mathcal{R}(u_{F_i} + \Delta u, \varphi_{F_i} + \Delta \varphi) = \mathcal{R}(u_{F_i}, \varphi_{F_i}) + \Delta u \left(\frac{\partial \mathcal{R}}{\partial u_{F_i}} \right) + \Delta \varphi \left(\frac{\partial \mathcal{R}}{\partial \varphi_{F_i}} \right) + \dots = 0 \quad (46)$$

respectively. Eqs. (45) and (46) can be written as

$$\left(\frac{\partial \mathcal{F}}{\partial u_{F_i}} \right) \Delta u + \left(\frac{\partial \mathcal{F}}{\partial \varphi_{F_i}} \right) \Delta \varphi = -\mathcal{F}(u_{F_i}, \varphi_{F_i}) \quad (47)$$

$$\left(\frac{\partial \mathcal{R}}{\partial u_{F_i}} \right) \Delta u + \left(\frac{\partial \mathcal{R}}{\partial \varphi_{F_i}} \right) \Delta \varphi = -\mathcal{R}(u_{F_i}, \varphi_{F_i}) \quad (48)$$

respectively, where Δu and $\Delta \varphi$ can be solved as

$$\Delta u = \frac{-\mathcal{F}(u_{F_i}, \varphi_{F_i}) \left(\frac{\partial \mathcal{R}}{\partial \varphi_{F_i}} \right) + \mathcal{R}(u_{F_i}, \varphi_{F_i}) \left(\frac{\partial \mathcal{F}}{\partial \varphi_{F_i}} \right)}{\left(\frac{\partial \mathcal{F}}{\partial u_{F_i}} \right) \left(\frac{\partial \mathcal{R}}{\partial \varphi_{F_i}} \right) - \left(\frac{\partial \mathcal{F}}{\partial \varphi_{F_i}} \right) \left(\frac{\partial \mathcal{R}}{\partial u_{F_i}} \right)} \quad (49)$$

$$\Delta \varphi = -\frac{-\mathcal{F}(u_{F_i}, \varphi_{F_i}) \left(\frac{\partial \mathcal{R}}{\partial u_{F_i}} \right) + \mathcal{R}(u_{F_i}, \varphi_{F_i}) \left(\frac{\partial \mathcal{F}}{\partial u_{F_i}} \right)}{\left(\frac{\partial \mathcal{F}}{\partial u_{F_i}} \right) \left(\frac{\partial \mathcal{R}}{\partial \varphi_{F_i}} \right) - \left(\frac{\partial \mathcal{F}}{\partial \varphi_{F_i}} \right) \left(\frac{\partial \mathcal{R}}{\partial u_{F_i}} \right)} \quad (50)$$

respectively, in which $\frac{\partial \mathcal{F}}{\partial u_{F_i}}$, $\frac{\partial \mathcal{R}}{\partial u_{F_i}}$, $\frac{\partial \mathcal{F}}{\partial \varphi_{F_i}}$, and $\frac{\partial \mathcal{R}}{\partial \varphi_{F_i}}$ can be represented as

$$\frac{\partial \mathcal{F}}{\partial u_{F_i}} = \frac{\mathcal{F}(u_{F_i} + \Delta u, \varphi_{F_i}) - \mathcal{F}(u_{F_i}, \varphi_{F_i})}{\Delta u} \quad (51)$$

Table 2Curve lengths of $\widehat{AC}$ and $\widehat{BD}$ and lengths of $\overline{AE}$ and $\overline{BF}$ for different polar angles ϑ and ϕ .

Right tooth surface			Left tooth surface		
$\vartheta / ^\circ$	$\widehat{AC} / \text{mm}$	$\overline{AE} / \text{mm}$	$\phi / ^\circ$	$\widehat{BD} / \text{mm}$	$\overline{BF} / \text{mm}$
23.377	6.077	5.711	60.443	6.078	5.711
24.277	5.318	4.997	59.172	5.318	4.997
25.180	4.559	4.284	57.923	4.558	4.283
26.086	3.799	3.570	56.694	3.798	3.569
26.995	3.039	2.856	55.488	3.038	2.856
27.908	2.279	2.142	54.299	2.279	2.142
28.824	1.519	1.427	53.130	1.519	1.427
29.744	0.760	0.714	51.978	0.760	0.714
31.599	0.760	0.714	49.725	0.759	0.713
32.534	1.519	1.427	48.620	1.519	1.427
33.475	2.279	2.142	47.531	2.279	2.142
34.422	3.039	2.856	46.455	3.039	2.856
35.376	3.798	3.569	45.393	3.799	3.570
36.337	4.558	4.283	44.344	4.559	4.284
37.305	5.317	4.996	43.308	5.318	4.997
38.282	6.076	5.710	42.282	6.078	5.711
39.268	6.836	6.424	41.267	6.837	6.425
40.262	7.595	7.137	40.262	7.579	7.122
41.267	8.355	7.851	39.268	8.356	7.852
42.282	9.115	8.565	38.282	9.116	8.566
43.308	9.875	9.279	37.305	9.876	9.280
44.344	10.634	9.993	36.337	10.635	9.994
45.393	11.393	10.706	35.376	11.394	10.707
46.455	12.153	11.420	34.422	12.154	11.421

$$\frac{\partial \mathcal{R}}{\partial u_{F\hat{i}}} = \frac{\mathcal{R}(u_{F\hat{i}} + \Delta u, \varphi_{F\hat{i}}) - \mathcal{R}(u_{F\hat{i}}, \varphi_{F\hat{i}})}{\Delta u} \quad (52)$$

$$\frac{\partial \mathcal{F}}{\partial \varphi_{F\hat{i}}} = \frac{\mathcal{F}(u_{F\hat{i}}, \varphi_{F\hat{i}} + \Delta \varphi) - \mathcal{F}(u_{F\hat{i}}, \varphi_{F\hat{i}})}{\Delta \varphi} \quad (53)$$

$$\frac{\partial \mathcal{R}}{\partial \varphi_{F\hat{i}}} = \frac{\mathcal{R}(u_{F\hat{i}}, \varphi_{F\hat{i}} + \Delta \varphi) - \mathcal{R}(u_{F\hat{i}}, \varphi_{F\hat{i}})}{\Delta \varphi} \quad (54)$$

respectively. Surface coordinates of any point of tooth surfaces of NG1 and NG2 can also be solved using the method mentioned above. The curve length between two neighboring points of tooth surfaces of NG1 and NG2 is equal to $p_n/24$ here. Coordinates of the gear tooth surface of NG1 can be calculated by Eqs. (45) and (46), as shown in Table 2.

5. Power loss evaluation of noncircular gears

Sliding friction power loss accounts for most of power losses of a gear system under low-speed and high-torque conditions [39,40]. Time-varying friction forces and sliding velocities of meshing tooth surfaces of noncircular gears are main excitation sources of sliding friction power loss.

5.1. Friction forces

Friction forces of gear tooth surfaces are determined by normal load distribution and the coefficient of friction (COF) of meshing tooth surfaces. In order to obtain normal load distribution of noncircular gears, a normal load distribution model is described by [30]

$$f(i_k, j) = \frac{\varepsilon \beta}{B} \frac{V_p(i_k, j)}{\int_{\ell} V_p(i_k, j) d\ell} F_N(i_k, j) \quad (55)$$

where (i_k, j) denotes the i th segment of the k th tooth of NG1 at the j th meshing instant, F_N is the normal load, $\varepsilon \beta$ is the face contact ratio, B is the face width, V_p is the inverse of the elastic potential energy, and ℓ is the length of each segment of NG1. More details are described in [30]. The friction force of the i th segment of the k th tooth of NG1 at the j th meshing instant can be represented as

$$F_{\mu}(i_k, j) = \zeta(i_k, j) f(i_k, j) \ell \quad (56)$$

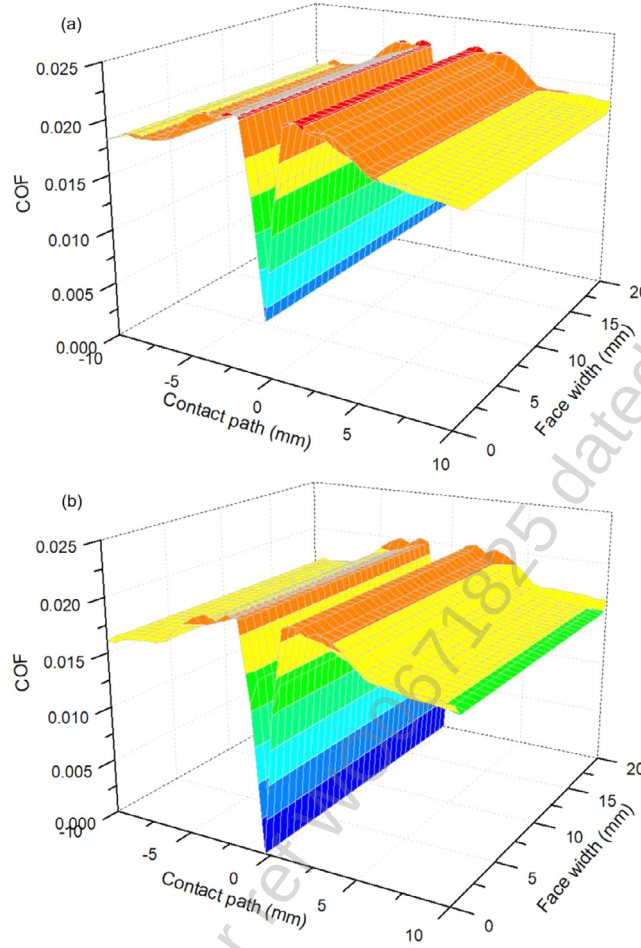


Fig. 6. Numerical results of COFs of tooth surfaces of the noncircular gear pair with (a) unmodified and (b) modified pitch curves.

where $\zeta(i_k, j)$ is the variable COF of a tooth surface that can be obtained by [38]

$$\zeta(i_k, j) = \frac{0.0127 \times 1.13}{1.13 - S_{avg}} \log_{10} \left[\frac{29700 f(i_k, j)}{\sigma v_s(i_k, j) v_e^2(i_k, j)} \right] \quad (57)$$

in which S_{avg} is the average surface roughness, v_s and v_e are the sliding velocity and entraining velocity, respectively, and σ is the viscosity of lubrication oil entering gear contact. The sliding velocity of the contact point changes with deviation of the curvature of the tooth surface due to modification of the pitch curve [41] and roughness of meshing tooth surfaces [31,42]. The sliding velocity $v_s(i_k, j)$ and entraining velocity $v_e(i_k, j)$ can be represented as

$$v_s(i_k, j) = |\mathbf{v}_{NG1}^r + \mathbf{v}_{NG1}^\omega - \mathbf{v}_{NG2}^r - \mathbf{v}_{NG2}^\omega| \quad (58)$$

$$v_e(i_k, j) = |\mathbf{v}_{NG1}^r + \mathbf{v}_{NG1}^\omega + \mathbf{v}_{NG2}^r + \mathbf{v}_{NG2}^\omega| \quad (59)$$

respectively, where $\mathbf{v}_{NG1}^r$ and $\mathbf{v}_{NG2}^r$ are moving velocities of a contact point of NG1 and NG2, respectively, and $\mathbf{v}_{NG1}^\omega = \omega_p \times \mathbf{r}_{NG1}$ and $\mathbf{v}_{NG2}^\omega = \omega_n \times \mathbf{r}_{NG2}$ are rotation velocities of the contact point around axes of NG1 and NG2, respectively. Numerical analysis settings of friction forces and power losses of the noncircular gear pair are listed in Table 3.

Numerical results of COFs of meshing tooth surfaces of the noncircular gear pair with unmodified and modified pitch curves are shown in Fig. 6(a) and (b), respectively. According to Eq. (57), the variable COF of a tooth surface is independent of the position of a contact point, and is relative to the sliding velocity $v_s(i_k, j)$ and entraining velocity $v_e(i_k, j)$ of the contact point of tooth surfaces. Based on numerical results of friction forces of tooth surfaces of the noncircular gear pair with unmodified and modified pitch curves in Fig. 6, the minimum COF of the tooth surface with the unmodified pitch curve is 4.8×10^{-3} , and that of the tooth surface with the modified pitch curve is 1.16×10^{-3} . The maximum COF of the tooth surface with the unmodified pitch curve is 0.02191, and that of the tooth surface with the modified pitch curve is 0.01987. According to Fig. 6(a) and (b), the minimum COFs of tooth surfaces with unmodified and modified pitch curves occur at

Table 3
Numerical analysis settings of friction forces and sliding friction power losses.

Item	Value
Input power (W)	750
Input load (Nm)	160
Rotational speed of NG1 (rpm)	45
Viscosity of lubrication oil (mm^2/s)	9.3
Average surface roughness	Ra3.6

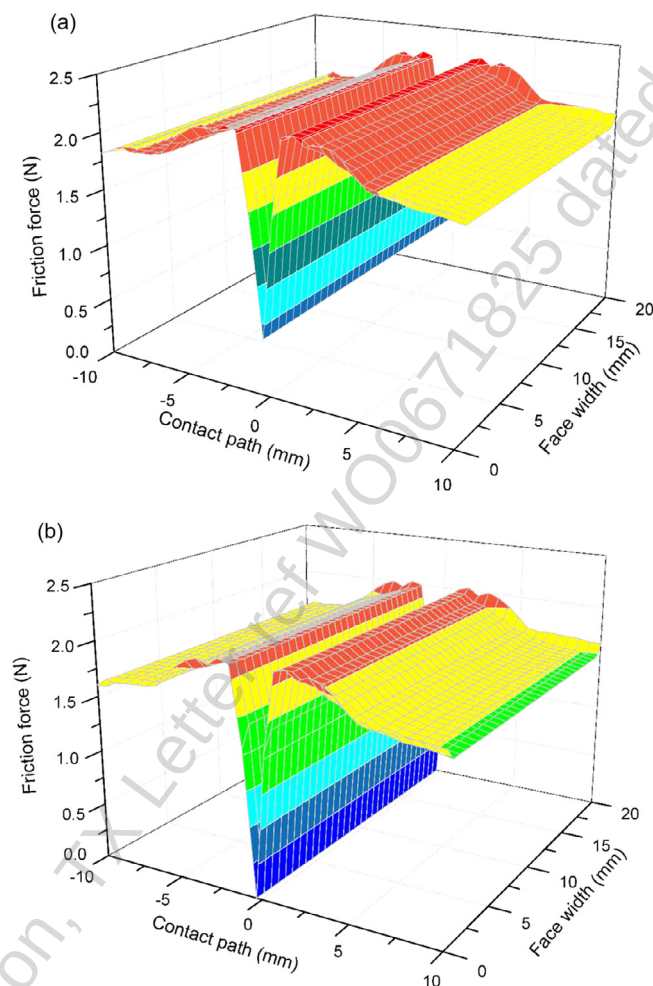


Fig. 7. Numerical results of friction forces of meshing tooth surfaces of the noncircular gear pair with (a) unmodified and (b) modified pitch curves.

pitch curves of noncircular gears. By comparing numerical results of friction forces of tooth surfaces of the noncircular gear pair with unmodified and modified pitch curves, the minimum and maximum COFs of the tooth surface with the modified pitch curve are smaller than those of the tooth surface with the unmodified pitch curve, respectively.

Based on numerical results of friction forces of meshing tooth surfaces of the noncircular gear pair with unmodified and modified pitch curves in Fig. 7, the minimum friction force of tooth surfaces with the unmodified pitch curve is 0.05 N, and that of tooth surfaces with the modified pitch curve is 0.012 N. The minimum friction forces of tooth surfaces with unmodified and modified pitch curves occur at their pitch points. The maximum friction force of tooth surfaces with the unmodified pitch curve is 2.202 N, and that of tooth surfaces with the modified pitch curve is 1.992 N. By comparing numerical results of friction forces of tooth surfaces of the noncircular gear pair with unmodified and modified pitch curves, the minimum and maximum friction forces of tooth surfaces with the modified pitch curve are smaller than those of tooth surfaces with the unmodified pitch curve, respectively. Modification of high-order pitch curves of noncircular gears can reduce friction forces of meshing tooth surfaces.

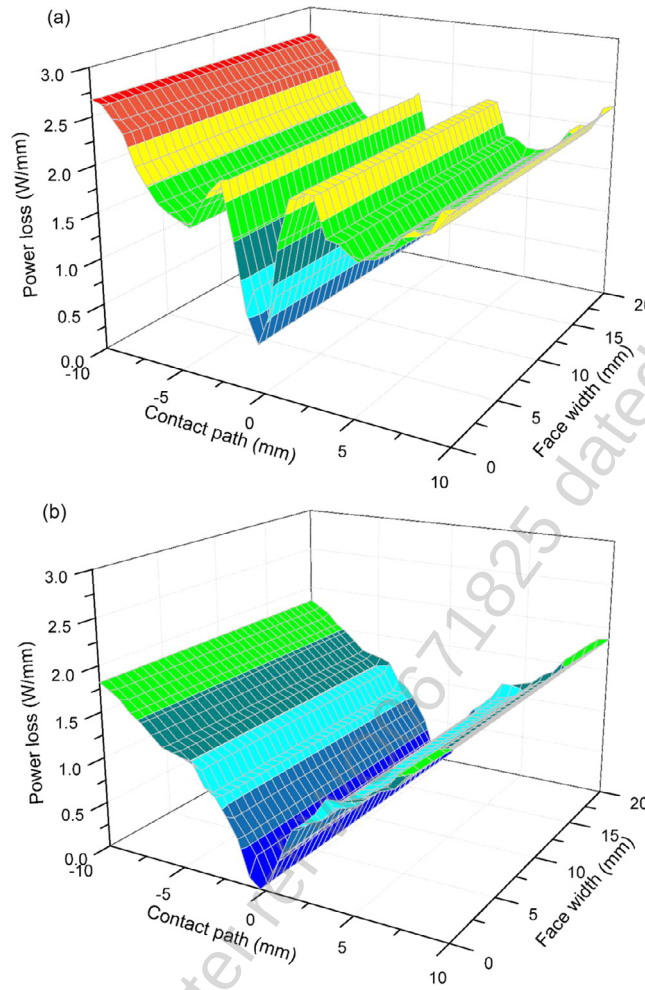


Fig. 8. Numerical results of sliding friction power losses of the noncircular gear pair with (a) unmodified and (b) modified pitch curves.

5.2. Sliding friction power loss

Since sliding friction power loss is determined by the work done by the friction force in a meshing period, meshing time is normalized with respect to a meshing period via a dimensionless variable $\tau = t/\Delta T$, where ΔT is the meshing period. By using the dimensionless time τ , the meshing period ΔT is equal to 1 and sliding friction power loss of the noncircular gear pair over one meshing period can be represented as

$$\begin{aligned} E_{friction} &= \int_{i_k} \int_j \int_0^1 F_{\mu}(i_k, j) r(j) \omega_p di_k dj d\tau \\ &= \int_{i_k} \int_j \int_0^1 \zeta(i_k, j) f(i_k, j) \ell v_s(i_k, j) di_k dj d\tau \end{aligned} \quad (60)$$

Numerical results of sliding friction power losses of meshing tooth surfaces of NG1 and NG2 with unmodified and modified pitch curves are shown in Fig. 8(a) and (b), respectively. According to Eq. (60), sliding friction power losses of meshing tooth surfaces depend on COFs, the normal load, and sliding velocities at a contact point of tooth surfaces of NG1 and NG2. The maximum sliding friction power losses of tooth surfaces with unmodified and modified pitch curves occur at dedenda of tooth surfaces. Sliding velocities at dedenda of tooth surfaces are larger than those at the pitch point of the tooth surface of NG1. Contact stresses at the dedendum of NG1 are also larger than those at the pitch point of its tooth surface. Larger friction forces with larger sliding velocities mean more sliding friction power loss at dedenda of tooth surfaces of NG1 and NG2.

The minimum sliding friction power losses of meshing tooth surfaces with unmodified and modified pitch curves occur at the pitch point of tooth surfaces. The maximum and minimum sliding friction power losses of the tooth surface with the modified pitch curve are 1.772 W/mm and 0.003 W/mm, respectively, which are smaller than those of the tooth surface with

the unmodified pitch curve, respectively. The maximum sliding friction power loss of the tooth surface with the unmodified pitch curve is 2.713 W/mm, and that of the tooth surface with the modified pitch curve decreases by 0.941 W/mm. The minimum sliding friction power loss of the tooth surface with the unmodified pitch curve is 0.481 W/mm, and that of the tooth surface with the modified pitch curve decreases by 0.478 W/mm. Average sliding friction power losses of tooth surfaces of NG1 and NG2 are significantly reduced due to modification of high-order pitch curves of noncircular gears.

6. Conclusions

This work presents a novel noncircular gear design method with a modified high-order elliptical pitch curve for the IVT to improve smoothness of pitch curves and manufacturability of noncircular gears. Modified high-order elliptical pitch curves of noncircular gears are designed based on kinematic equations of the IVT. Sliding friction power losses of the noncircular gear pair with unmodified and modified pitch curves are evaluated by incorporating load-dependent friction forces and sliding velocities. Based on numerical results obtained, some meaningful conclusions are drawn below.

- (1) The proposed noncircular gear design method can in principle be used for determining proper meshing performance of gear transmissions with noncircular gears. While this study is associated with investigation of the IVT, the method can be used for noncircular gears in other types of gear transmissions.
- (2) Noncircular gears with high-order elliptical pitch curves can be fabricated and modified by using a shaping cutter. Modification of high-order elliptical pitch curves can guarantee positive convexities of the whole high-order elliptical pitch curves and eliminate cusps between every two neighboring lobes of high-order elliptical pitch curves.
- (3) Sliding friction power losses of meshing tooth surfaces of noncircular gears can be estimated based on sliding velocities and friction forces of tooth surfaces. Numerical analysis results show that modification of high-order elliptical pitch curves can reduce sliding friction power losses of tooth surfaces.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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